where $\det[\Gamma(s)] = \det(sI_{2n1+q} - \tilde{H}_{11})\det(sI_\infty - A_r)$, $\Delta H_r(s)$ and $\Delta H_q(s)$ are defined as indicated in Eq. (19). By assumptions we can design the gain/filter matrices (G,K) and (C_q, A_q, B_q) such that $\det(sI_{2n1+q} - \tilde{H}_{11}) \neq 0$, $\forall s \in \mathcal{C}_+$. Furthermore, $\det(sI_\infty - A_r) \neq 0$, $\forall s \in \mathcal{C}_+$. Therefore, the stability property is uniquely determined by the third determinant in the final equality in the preceding. Also note that for all $(B,C) \in \mathfrak{X}$ and $\Delta H_q(s) \in \mathcal{S}^{r\times r}$, $\|\Delta H_r(s)\Delta H_q(s)\| \leq \|\Delta H_r(s)\| \, \|\Delta H_q(s)\| < \infty$, $\forall s \in \mathcal{C}_+$. By the same argument for concluding the loop stability as in Theorem 1, the closed-loop system (18) will then be stabilized by the proposed RMF provided that inequality (19) holds. This completes the proof.

Note that removing the RMF is equivalent to letting $\Delta H_q(s) = I_r$. For this situation the result of Theorem 3 reduces to Eq. (15).

The robust design principle is that the RMF cutoff frequency should be placed above the highest controlled mode natural frequency to reduce the filtering effect on the controlled modes, and as far below the residual modes as possible to maximize attenuation of the residual mode response. That is, design the RMF matrices (A_q, B_q, C_q) such that $\Delta H_q(s)$ exhibits a low-pass property so that the perturbation bound $\|\Delta H_q(j\omega)\Delta H_r(j\omega)\|$ is minimized over a high-frequency band, say $[\omega_c, \infty]$. We can, for example, design the RMF to meet

$$\|\Delta H_q(j\omega)\| : \approx 1 \text{ if } |\omega| \leq 10\omega_c$$

$$: \approx 0 \text{ if } |\omega| > 100\omega_c$$

Then proceed to design the observer-based controller (17) such that inequality (19) holds.

Both types of the RMF designs presented here minimize spillover's impact on the loop stability. However, comparing the operation of a parallel RMF with a cascaded one, in the former one needs to know about the residual dynamics precisely to approximately cancel the critical residual modes; whereas, for the latter, only a roughly estimated cutoff frequency for the controlled modes is required to implement the filter. For the controller synthesis, the former is totally independent with the observer design; whereas, for the latter, the RMF is included in the observer and hence complicates the nominal control design.

Conclusions

This Note presents a method for developing robust design criteria for active control of flexible systems subjected to unmodeled residual modes. It is proved that asymptotic loop stability can be achieved by the addition of a parallel or cascaded connected residual mode filter with an observer. It is also shown that the additional filter can provide excessive robust stability margins and hence alleviates the impact from the spillover to stability.

References

[1]Balas, M. J., "Feedback Control of Flexible Systems," *IEEE Transactions on Automatic Control*, Vol. AC-23, No. 8, 1978, pp. 673–679.

[2]Balas, M. J., Quan, R., Davidson, R., and Das, B., "Low-Order Control of Large Aerospace Structures Using Residual Mode Filters," *Recent Advances in Control of Nonlinear and Distributed Parameter Systems, Robust Control and Aerospace Applications*, American Society of Mechanical Engineers, New York, 1988, pp. 157–168.

[3]Chait, Y., and Radcliffe, C. J., "Control of Flexible Structures with Spillover Using an Augmented Observer," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 1, 1989, pp. 155–161.

[4]Sakawa, Y., "Feedback Control of Second-Order Evolution Equations with Damping," *SIAM Journal of Control and Optimization*, Vol. 22, No. 3, 1984, pp. 343–361.

[5]Wie, B., and Byun, K.-M., "New Generalized Structural Filtering Concept for Active Vibration Control Synthesis," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 1, 1989, pp. 147–154.

[6]Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1976.

[7]Vidyasagar, M., *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985.

[8]Skelton, R. E., *Dynamic Systems Control*, Wiley, New York, 1988.

[9]Tang, T. S., and Hung, G., "On Control of Large Space Structures," *Proceedings of the 1990 American Control Conference*, San Diego, CA, May 1990, pp. 2965–2970.

Buckling Control of a Flexible Beam Using Piezoelectric Actuators

Tesfay Meressi* and Brad Paden†
University of California, Santa Barbara, Santa Barbara, California 93106

Introduction

ACTIVE damping and control of flexible structures has been an area of research focus for some time.[1] However, the recent application of distributed piezoelectric actuators to structure control by Crawley and De Luis[2] and Bailey and Hubbard[3] has posed new and challenging problems. Following the initial experiments of these researchers, where a single vibrational mode is controlled, Fisher[4] addressed the actuator placement problem to control several modes.

In this Note we address the new problem of buckling control using smart materials. In contrast to the dynamic stability issues of vibration control, buckling is a static instability of axially loaded members of a structure. It is well known that as the axial compressive load P in an initially straight beam increases, the beam remains straight and undeformed until the load reaches a certain critical value $P_{cr,1}$, where the stable equilibrium of the first bending mode bifurcates into one unstable and two stable equilibria (pitchfork bifurcation). The two stable equilibria correspond to buckled configurations.

Here we use piezoelectric actuators and strain gauge sensors to show that buckling of a simply supported beam can be postponed beyond the first critical load. The load deflection characteristic for large deflections of a beam in a buckled configuration is highly nonlinear and involves numerical solution of elliptic integrals. Figure 1a shows a typical load deflection curve where $P_{cr,n}$ is the buckling load of the nth mode. If $P < P_{cr,1}$, the undeflected beam is stable. For $P_{cr,n} < P < P_{cr,n+1}$ all modes are stable except for the first n bending modes. The idea reflected in this Note is the use of feedback control in conjunction with piezoactuators to stabilize the first bending mode beyond $P_{cr,1}$ and achieve a bifurcation diagram of the form shown in Fig. 1b, where the buckling force $P_{cr,1}$ is greater than that for the uncontrolled beam.

We begin by deriving the linearized equation of motion and the associated modal equations of a simply supported flexible beam with piezoelectric actuators subjected to slowly varying axial load. This is followed by the state-space model of the reduced order system and the design of a controller to increase the stiffness or impedance of the first bending mode. We discuss the effect of the unmodeled higher order residual modes and methods of reducing this effect. Our conclusions are made in the last section.

Received Feb. 25, 1992; revision received Oct. 10, 1992; accepted for publication Oct. 19, 1992.

*Research Assistant, Department of Mechanical Engineering.

†Associate Professor, Department of Mechanical Engineering.

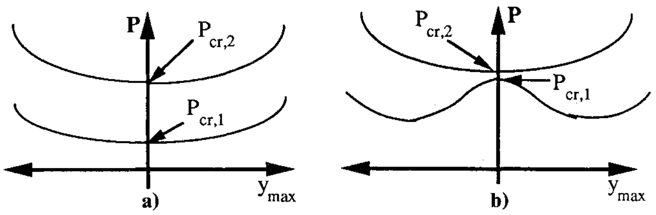

Fig. 1 Load deflection curve of a) uncontrolled beam and b) controlled beam.

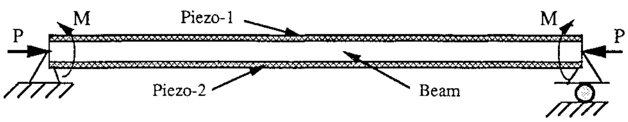

Fig. 2 Simply supported column with piezoelectric actuators.

System Model of a Beam with Piezoactuators

In this section we use a truncated modal expansion of the deflection of a beam to derive a linear finite-dimensional model. We emphasize that the beam is assumed to be uniform with no manufacturing imperfections. Since the aim is to stabilize the beam in its straight configuration, it is natural to assume small deflections and linearize the equations of motion about this configuration. Note that the strain induced by piezoelectric actuators is usually small and, therefore, the small deflection assumption is consistent with the capacity of the actuators.

Figure 2 shows a simply supported uniform beam with piezoelectric actuators of equal thickness bonded to both sides by a suitable adhesive. The beam of width b and thickness t_b is subjected to an axial compressive load, and control moments are applied by the piezoactuators. The actuator being modeled is a piezoelectric polymer, poly vinylidene fluoride. For an axially polarized piezo, a voltage applied across its thickness results in strain along its length. For simplicity the width of each piezolayer is assumed to be the same as that of the beam.

The strain Λ_i developed in an unconstrained piezo is given by $\Lambda_i = \nu_i(t)d_{31}/t_p$ where $\nu_i(t)$; $i = 1, 2$ is the voltage applied to the ith piezostrip, d_{31} the piezoelectric strain constant, and t_p the thickness of the piezolayer. If ν_1 and ν_2 are the voltages applied on the top and bottom piezolayers, respectively, and E_p is the Young's modulus of the piezo, the resulting moment on the piezobeam segment is given by

$$M = bE_p t_p(\Lambda_2 - \Lambda_1)\left(\frac{t_b}{2} + t_a + \frac{t_p}{2}\right)$$

$$= bE_p d_{31}\left(\frac{t_b}{2} + t_a + \frac{t_p}{2}\right)(\nu_2 - \nu_1) \triangleq k^*(\nu_2 - \nu_1) \tag{1}$$

The equation of motion of the beam can be derived using Hamilton's variational principle.[5] Under small deflection assumption Hamilton's principle yields

$$\rho A\ddot{y} + (EIy'')'' + (Py')' = M[\delta'(x - x_2) - \delta'(x - x_1)] \tag{2}$$

where ρ is the density of the beam; y is the transverse deflection; $\dot{y}$ and y' are the time and spatial derivatives of y, respectively; EI is the stiffness of the beam; A is the cross-sectional area of the beam; δ' is the spatial derivative of the delta function; and x_1 and x_2 are the locations of the two ends of the piezolayer. The solution to Eq. (2) is obtained as follows.

Unforced Dynamics

The unforced system dynamics are defined by the condition $M = 0$ in Eq. (2). Using separation of variables, we have $y_n(x,t) = \phi_n(x)\eta_n(t)$. Substituting this in Eq. (2) and assuming a simply supported uniform beam (EI constant) of length L, we obtain

$$\eta_n(t) = \sin(\omega_n t + \psi) \quad \text{and} \quad \phi_n(x) = D_n \sin(n\pi x/L)$$
$$n = 1, 2, 3 \ldots \tag{3}$$

If ϕ_n is normalized so that $\int_0^L \phi_n^2 \, dx = 1$, the unforced dynamics in modal form becomes

$$y_n(x,t) = \sqrt{2/L}\,\sin(n\pi x/L)\sin(\omega_n t + \psi) \tag{4}$$

To see how the load affects the natural frequencies of the beam, substitute Eq. (4) in Eq. (2) and simplify to get

$$\omega_n^2 = \frac{1}{\rho A}\frac{n^2\pi^2}{L^2}\left[EI\frac{n^2\pi^2}{L^2} - P\right] \tag{5}$$

From Eq. (5) we see that the nth pole pair $\pm j\omega_n$ moves along the imaginary axis toward the origin as P increases from 0 to $P_{cr,n} \triangleq EIn^2\pi^2/L^2$ and becomes real for $P \geq P_{cr,n}$. Hence, when $P = P_{cr,n}$ the beam loses all of its stiffness and buckles in the nth buckling mode.

Forced Response

The deflection of the beam can be written in terms of the modal deflections as

$$y(x,t) = \sum_{n=1}^{n=\infty} y_n(x,t)$$

Substituting this in Eq. (2) we obtain

$$\sum_{n=1}^{n=\infty} \rho A\phi_n\ddot{\eta}_n + \sum_{n=1}^{n=\infty} EI\phi_n''''\eta_n + \sum_{n=1}^{n=\infty} P\eta_n\phi_n''$$
$$= M[\delta'(x - x_2) - \delta'(x - x_1)] \tag{6}$$

(If the piezoelectric actuators are bonded along the total length of the beam, $x_1 = 0$ and $x_2 = L$.) Multiplying both sides of Eq. (6) by ϕ_n, integrating with respect to x over the beam length, and using the orthogonality of the mode shapes yields

$$\ddot{\eta}_n + \omega_n^2\eta_n = \frac{-2n\pi M}{\rho AL}\sqrt{\frac{2}{L}}\begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{7}$$

If $\nu_2 = -\nu_1 \triangleq \nu$, we have $M = 2k^*\nu$ and Eq. (7) becomes $\ddot{\eta}_n + \omega_n^2\eta_n = B_n\nu$, where

$$B_n \triangleq \frac{-4n\pi k^*}{\rho AL}\sqrt{\frac{2}{L}}\begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{8}$$

Since the n even modes are uncontrollable, we expect the beam to buckle in the second mode when $P \geq P_{cr,2}$.

Many control problems aimed at vibration suppression use piezoelectric materials as sensors in the feedback loop. However, due to charge leakage, piezoelectrics are not useful as sensors near 0 Hz as required in this application. Therefore, resistive strain gauge sensors are used as modeled in the next section.

Sensor Modeling

Modal states are estimated from strain gauge measurements at discrete locations. It is easy to see that observability of the modes of the system depends on the location of the sensors; a mode with its node at the location of a strain gauge is unobservable with that sensor. To reduce the number of sensors, modal control of flexible structures is usually based on the first few modes of vibration. This is justified by the fact that higher vibrational modes are in general difficult to excite and have higher structural damping. However, the unmodeled dynamics can cause instability through what are known as control and observation spillover. It has been shown[1] that both control and observation spillover of unmodeled modes are necessary to cause instability in a closed-loop system.

The sensors are placed so that the second and third modes and their multiples are unobservable. Thus the first and fifth modes are the first two modes in a minimal realization of the system. We ignore higher order modes and discuss the associated spillover problems later. If a small amount of structural damping is present, all of the unobservable modes remain stable even in the presence of spillover. Similarly the dynamics of the even modes are not affected by the control, and hence they remain stable.

We model the sensors as follows. The bending moment at the location of a strain gauge, a distance x from the left end of the beam, is given by[5]

$$M_b = -E_b I_{eq} y''(x,t) = \sqrt{\frac{2}{L}}\, E_b I_{eq} \frac{\pi^2}{L^2} \sum_{n=1}^{n=\infty} n^2 \eta_n \sin\left(\frac{n\pi x}{L}\right) \tag{9}$$

where E_b is the Young's modulus of the beam material and I_{eq} is the equivalent moment of inertia of the composite piezo-beam segment based on beam material. The resulting strain in a strain gauge attached to one side of the beam is

$$\epsilon = \pm \frac{M_b\left[(t_b/2) + t_a + t_p\right]}{E_b I_{eq}} - \frac{P}{A_{eq} E_b} \tag{10}$$

where A_{eq} is the equivalent area based on beam material. The sign of the first term in Eq. (10) depends on which side of the beam the strain gauge is bonded to. Therefore, the output of a *differential* strain gauge is *independent of the load P* and is given by

$$\nu_s = 2\epsilon k_g = 2\sqrt{\frac{2}{L}}\, k_g \frac{\left[(t_b/2) + t_a + t_p\right]\pi^2}{L^2} \sum_{n=1}^{n=\infty} \times n^2 \eta_n \sin\left(\frac{n\pi x}{L}\right) \triangleq k_s \sum_{n=1}^{n=\infty} n^2 \eta_n \sin\left(\frac{n\pi x}{L}\right) \tag{11}$$

where k_g is the strain gauge constant. If differential strain gauges are placed at $x = L/3$ and $x = 2L/3$, and the sum of their measurements is taken as the system output, we have

$$\nu_o \triangleq \nu_s\left(x = \frac{L}{3}\right) + \nu_s\left(x = \frac{2L}{3}\right) = k_s \sum_{n=1}^{n=\infty} n^2 \eta_n \left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right) \tag{12}$$

Controller Design

If n modal amplitudes and their rates are taken as the states of the system, the state-space representation of the $2n$ dimensional reduced order model with a state vector $\eta_r^T = [\eta_1 \quad \dot\eta_1 \quad \cdots \quad \eta_n \quad \dot\eta_n]$ becomes

$$\dot\eta_r = A_r \eta_r + B_r \nu \qquad \text{and} \qquad V_o = C_r \eta_r \tag{13}$$

where A_r is a $2n \times 2n$ block diagonal matrix, B_r is a $2n \times 1$ input matrix, and ν is input to the system. We assume that the derivative of the sensor output can be computed and define $V_o^T \triangleq [\nu_o \quad \dot\nu_o]$; and C_r is a $2 \times 2n$ output matrix. If only the first and the fifth modes are included in the reduced order model and a structural damping coefficient ζ is assumed, then

$$A_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\zeta\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_5^2 & -2\zeta\omega_5 \end{bmatrix}$$

$$B_r = \frac{-4\pi k^*}{\rho A L}\sqrt{\frac{2}{L}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix}, \qquad C_r^T = k_s \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \\ -25\sqrt{3} & 0 \\ 0 & -25\sqrt{3} \end{bmatrix} \tag{14}$$

A controller is designed using standard linear quadratic regulator (LQR) design to minimize a cost functional of the form

$$J = \int_0^\infty (\alpha \eta_r^T Q \eta_r + \nu^T R \nu)\, dt \tag{15}$$

where Q and R are positive semidefinite and positive definite weighting matrices, respectively; and α is a scalar. The solution of the corresponding Riccati equation in this method gives an optimal state feedback solution of the form $\nu(t) = -K_r \eta_r(t)$, where K_r is a constant feedback gain matrix. The following parameters were used for simulation.

Beam properties:

$$b_b = 25.4 \text{ mm} \qquad t_b = 1 \text{ mm} \qquad E_b = 5 \text{ GPa}$$

$$\rho_b = 1000 \text{ kg/m}^3 \qquad L_b = 152.4 \text{ mm}$$

Piezoactuaor properties:

$$b_p = b_b \qquad t_p = 110\ \mu\text{m} \qquad E_p = 2 \text{ GPa} \qquad L_p = L_b$$

$$\rho_p = 1780 \text{ kg/m}^3 \qquad d_{31} = 23 \times 10^{-12} \text{ m/V}$$

A strain gauge constant $k_g = 0.01$ V/μ *strain* and structural damping coefficient $\zeta = 0.01$ are assumed. The thickness of the adhesive layer is neglected.

The optimal feedback gain matrix K_r is computed for $P = 4.1 P_{cr,1}$ using $Q = C_r^T C_r$, $R = 1$, and $\alpha = 0.05$. The first mode is stabilized at a load exceeding $P_{cr,2} = 4P_{cr,1}$ and at this load the uncontrolled second mode is unstable (i.e., the beam is forced to buckle in the second mode). However, it remains to check that this controller stabilizes the system for loads less than $4.1P_{cr,1}$. For the reduced order system this problem can be reduced to checking the roots of a fourth-degree interval polynomial with each coefficient varying monotonically when the load varies from $P = 0$ to $P = 4.1P_{cr,1}$. The stability of the system under any fixed axial load $P \leq 4.1P_{cr,1}$ was verified by checking the stability of the two Kharitonov polynomials[6] associated with the fourth-order characteristic equation of the system. The same robustness result can also be obtained numerically using a root locus plot parameterized by P.

The resulting closed-loop response to nonzero initial conditions and the control input voltage to the actuators for the controlled model with a load of $P = 3.8P_{cr,1}$ are shown in Fig. 3a. The effect of the unmodeled dynamics and methods of reducing this effect is the subject of the next section.

Numerical Evaluation of Spillover

To see the effect of spillover, the same gain K_r is used with an extended evaluation model containing modes 7 and 11 in addition to modes 1 and 5. To reduce the effect of spillover the

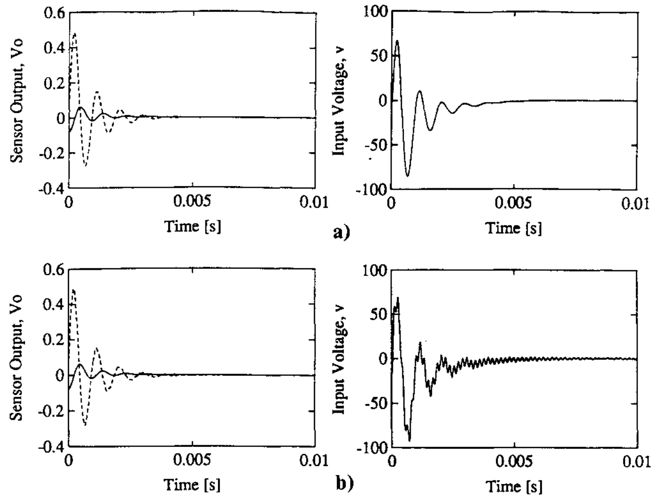

Fig. 3 Closed-loop response to nonzero initial deflection ($P = 3.8P_{cr,1}$): a) reduced order model and b) extended evaluation model.

controlled modes are reconstructed from the output of the extended model using a comb filter selecting modes 1 and 5 (an observer serves this purpose) and these estimates are used for feedback.

The resulting closed-loop response of the augmented system to nonzero initial conditions and the required input voltage to the actuators are shown in Fig. 3b. From Figs. 3a and 3b it can be seen, as expected, that there is no significant effect of the uncontrolled modes on the dynamics of the controlled modes.

Conclusion

In this Note we addressed the problem of buckling control using smart materials, a static instability of axially loaded members of a structure. We showed that the buckling of a flexible beam can be postponed beyond the first critical load by means of feedback using piezoelectric actuators and strain gauge sensors. It is observed that a controller design based on a fixed axial load P_{max} stabilizes the modeled modes for any $P \leq P_{max}$ and, therefore, is robust to slow load variations. Hence buckling in the first mode is inhibited, and the beam can support a load up to the second critical load. Actuator and sensor placement is discussed with regard to problems of spillover. Finally, spillover has not posed serious problems as we are able to design the controller, in the case of a beam, using a low-order model and verify stability for a high-order model.

Acknowledgment

Support from the Air Force Office of Scientific Research through Grant F49620251-C-0095 is gratefully acknowledged.

References

[1]Balas, M. J., "Active Control of Flexible Systems," *Journal of Optimization Theory and Applications*, Vol. 25, No. 3, 1978, pp. 415–436.

[2]Crawley, E. F., and De Luis, J., "Use of Piezoelectric Actuators as Elements of Intelligent Structures," *AIAA Journal*, Vol. 25, No. 10, 1987, pp. 1373–1385.

[3]Bailey, T., and Hubbard, J. E., "Distributed Piezoelectric-Polymer Active Vibration Control of a Cantilever Beam," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 5, 1985, pp. 605–611.

[4]Fisher, S., "Application of Actuators to Control Beam Flexure in a Large Space Structure," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 6, 1989, pp. 874–879.

[5]Timoshenko, S. P., and Gere, J. M., *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.

[6]Minnichelli, R. J., Anagnost, J. J., and Desoer, C. A., "An Elementary Proof of Kharitonov's Stability Theorem with Extensions," *IEEE Transactions on Automatic Control*, Vol. AC-34, Sept. 1989, pp. 995–998.

Robust Time-Optimal Control of Uncertain Structural Dynamic Systems

Bong Wie,* Ravi Sinha,† and Qiang Liu†
Arizona State University, Tempe, Arizona 85287

Introduction

THE problem of computing robust time-optimal control inputs for uncertain flexible spacecraft has been recently investigated in Refs. 1 and 2. The primary control objective in such a robust control problem is to achieve a fast maneuvering time with minimum structural vibrations during and/or after a maneuver in the face of modeling uncertainty. In this Note we treat a similar problem to explore the effects of using a pair of noncollocated, one-sided jets. A parameter optimization approach, with additional constraints for performance robustness with respect to modeling uncertainty, is employed to solve such an interesting control problem.

A simple mathematical model of an uncertain structural dynamic system with one rigid-body mode and two flexible modes, as shown in Fig. 1, is used to illustrate the concept and methodology. Such a simple model is often used to represent an actual spacecraft with few dominant structural modes for the purposes of practical control design.[3,4] We consider a special case in which the structural flexibility and mass distribution of the system are quite uncertain, although the total mass (or inertia) of the system is well known. Consequently, we focus on the robust control problem of flexible structures in the face of modal frequency uncertainty as well as mode shape uncertainty. However, many theoretical issues inherent to constrained parameter optimization problems and the practical implementation issue inherent to any open-loop control approach are not discussed in this Note.

Problem Formulation

Consider the mass-spring model shown in Fig. 1, which is, in fact, a generic representation of a spacecraft with one rigid-body mode and two flexible modes. The modal equations of this system can be represented as

$$\ddot{y}_1 = \phi_{11}u_1 + \phi_{12}u_2 + \phi_{13}u_3 \tag{1a}$$

$$\ddot{y}_2 + \omega_2^2 y_2 = \phi_{21}u_1 + \phi_{22}u_2 + \phi_{23}u_3 \tag{1b}$$

$$\ddot{y}_3 + \omega_3^2 y_3 = \phi_{31}u_1 + \phi_{32}u_2 + \phi_{33}u_3 \tag{1c}$$

where y_i is the ith modal coordinate, ω_i the ith modal frequency, ϕ_{ij} the modal input distribution coefficients, and u_i the control inputs. The nominal parameter values are assumed as $m_1 = m_2 = m_3 = k_1 = k_2 = 1$ with appropriate units, and time is in units of second.

Using this simple generic model, we explore three cases: 1) case 1 with both "positive" and "negative" jets placed at body 1, 2) case 2 with a positive jet at body 1 and a negative jet at body 2, and 3) case 3 with a positive jet at body 1 and a negative jet at body 3. Case 1 is a typical case in which two opposing jets are collocated. In cases 2 and 3, two opposing jets are not collocated. In this Note, we consider only case 3, whereas detailed results for cases 1 and 2 can be found in Ref. 5.

Received March 11, 1992; revision received Dec. 1, 1992; accepted for publication Dec. 6, 1992.

*Professor, Department of Mechanical and Aerospace Engineering. Associate Fellow AIAA.

†Graduate Research Assistant, Department of Mechanical and Aerospace Engineering. Student Member AIAA.